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Approximate Controllability of Semilinear Integro-Differential Fractional Control Systems with Nonlocal Conditions

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Abstract

The generic aim of this paper is to provide a new set of sufficient conditions of approximate controllability for semilinear integro-differential fractional control systems with nonlocal conditions in a Banach space X. The results are obtained with the help of Schauder fixed point theorem and compact semigroup operator.

Keywords: Approximate controllability, semilinear fractional system, semigroup theory, fixed point theorem

1 Introduction

A controllability of systems has been playing an important role in many academic fields and applied sciences such as, engineering, physics, biology, and finances. At a glance, controllability is the idea of steering to any state value at the final time from the initial state value. However, since the conditions for exact controllability are too strong, it is quite reasonable to generate a new concept of controllability with weaker conditions called approximate controllability (see Fattorini [6] and Russel [22]). They are both adequately discussed and represented in, for example, Curtain and Zwart [5], Bensoussan et al. [4], Zabczyk [25], Klamka [10] etc. for deterministic linear systems, and Klamka [11], Li and Yong [14] etc. for deterministic nonlinear systems.

In recent decades, many problems which have been emerging in various applied fields such as, engineering, physics, biology, chemistry, aerodynamics and finances can be characterized by fractional differential equations. For more details, we refer the readers to the monographs [21, 12, 18] and references therein. Motivated by these, several authors have committed to extend the concept of controllability to fractional systems. Approximate and complete Controllability for linear fractional systems have been adequately studied. However, For semilinear fractional systems, the present situation is distinct and not satisfactory. Wang and Zhou [23] studied the exact controllability of fractional evolution systems without assuming the compactness of semigroup operator. Also, in [24] they showed the existence and controllability results for fractional semilinear differential inclusions by using Bohnenblust-Karlin's fixed point theorem. Kamaljeet and Bahuguna [9] established a set of sufficient conditions for the approximate controllability of nonlocal neutral fractional integro-differential equations with finite delay in Hilber space, the results are obtained by using fractional differential equations, analytic semigroups and Krasnoselskii's fixed point theorem. Mahmudov and Zorlu [15] discussed the approximate controllability of fractional evolution equations with compact analytic semigroup and the results are given by employing fixed point technique with the assumption that the associated linear system is approximately controllable.

In this paper, our central goal is to provide a new set of sufficient conditions of approximate controllability of the semilinear fractional control system with nonlocal conditions of the form

$$\begin{cases} D^q[x(t) - \Lambda_k(x)(t)] = A[x(t) - \Lambda_k(x)(t)] + f(t, x(t)) + Bu(t),\ 0 < t \leq b, \\ x(0) = x_0 + g(x), \end{cases}$$

assuming that its associated linear part is approximately controllable for any positive real $\epsilon > 0$. The technique used here was applied by Bashirov and Ghahramanlou [1] for deriving sufficient conditions for approximate controllability of semilinear systems, Leiva [13] for proving the approximate controllability of semilinear impulsive evolution equations, and Guevara and Leiva [7] for establishing the approximate controllability of the impulsive semilinear heat equation with memory and delay. The essential feature of this technique that approximate controllability for the semilinear systems obtained without assuming the approximate controllability for the associated linear system.

The layout of this paper is as follows: used notations are presented in second section. In section 3, we obtain main assumptions and sufficient conditions of approximate controllability for the above control system. A brief conclusion is given in the last section.

2 Preliminaries

Throughout this paper, all the integrals will be taken in Bochner's sense. in this section, we collect some well-known definitions and results in fractional differential theory. In addition we give the needful notation which will be utilized in this paper.

Definition 1. *[21]* The Riemann-Liouville fractional integral of an integrable function f of any arbitrary order $q > 0$ is defined as

$$I_0^q f(t) = \frac{1}{\Gamma(q)} \int_0^t f(s) \frac{1}{(t-s)^{1-q}}\, ds, \quad q > 0$$

where $\Gamma(q) = \int_0^\infty s^{q-1} e^{-s}\, ds$ is the gamma function at point q.

Definition 2. *[21]* The Riemann-Liouville fractional derivative of order $q > 0$ of a suitable function f is defined as

$$D_0^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t f(s) \frac{1}{(t-s)^{1+q-n}}\, ds, \quad q \in (n-1,\, n].$$

Definition 3. *[21]* The Caputo fractional derivative of order $q \in (n-1,\, n]$ of a suitable function f is defined as

$$D^q f(t) = I_0^{n-q} f^{(n)}(t) = \frac{1}{\Gamma(n-q)} \int_0^t f^{(n)}(s) \frac{1}{(t-s)^{1+q-n}}\, ds, .$$

In particular, if $0 < q \leq 1$, then

$$D^q f(t) = I_0^{1-q} f^{(')}(t) = \frac{1}{\Gamma(1-q)} \int_0^t f^{'}(s) \frac{1}{(t-s)^{1-q}}\, ds, .$$

Lemma 1. The Laplace transform of Caputo fractional derivative of order $q \in (n-1,\, n]$ can be given as

$$L[D^q f(t)] = s^q F(s) - \sum_{k=0}^{n-1} f^k(0) s^{q-1-k},$$

and for $0 < q \leq 1$, we have

$$L[D^q f(t)] = s^q F(s) - f(0) s^{q-1},$$

where $F(s) = \int_0^\infty e^{-st} f(t)\, dt$ is tha Laplace transform of $f(t)$.

Consider the following integro-differential fractional control system with nonlocal conditions

$$\begin{cases} D^q[x(t) - \Lambda_k(x)(t)] = A[x(t) - \Lambda_k(x)(t)] + f(t, x(t)) + Bu(t),\ 0 < t \leq b, \\ x(0) = x_0 + g(x), \end{cases} \tag{1}$$

where D^q is the derivative operator of order $0 < q \leq 1$ in sense of Caputo as defined above, x is the state variable belongs to the Banach space X.
Applying Riemann-Liouville fractional integral on both sides of system (1), we get

$$\begin{aligned} x(t) - \Lambda_k(x)(t) = x_0 + g(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A[x(s) - \Lambda_k(x)(s)]\, ds \\ + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s, x(s)) + Bu(s)]\, ds, \end{aligned} \tag{2}$$

provided that all the integrals exist.
Now, applying the Laplace transformation to both sides of this equation, we obtain

$$\begin{aligned} L\{x(t) - \Lambda_k(x)(t)\} &= \frac{1}{\mu}[x_0 + g(x)] + \frac{1}{\mu^q} AL\{x(t) - \Lambda_k(x)(t)\} \\ &\quad + \frac{1}{\mu^q} L\{f(t, x(t)) + Bu(t)\} \\ &= \frac{\mu^{q-1}}{(\mu^q I - A)}[x_0 + g(x)] + \frac{1}{(\mu^q I - A)} L\{f(t, x(t)) + Bu(t)\}. \end{aligned} \tag{3}$$

we have $\frac{1}{(\mu^q I - A)}(x) = \int_0^\infty e^{-\mu^q s} T(s)x\, ds, x \in X$, (see Pazy [19]) where I is the identity operator defined on X.
Substituting this identity into above equation, we get

$$\begin{aligned} L\{x(t) - \Lambda_k(x)(t)\} &= \mu^{q-1} \int_0^\infty e^{-\mu^q s} T(s)[x_0 + g(x)]\, ds \\ &\quad + \int_0^\infty e^{-\mu^q s} T(s) L\{f(t, x(t)) + Bu(t)\}\, ds \\ &= J_1 + J_2, \end{aligned} \tag{4}$$

providing that all these integrals exist.
Define the one-side stable probability density as

$$\phi(v) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} v^{-kq-1} \frac{\Gamma(qk+1)}{n!} sin(qk\pi)\ \ v \in (0, \infty)$$

and its Laplace transform $L\{\phi(v)\} = e^{-\mu^2}$.
Now, using $e^{-(\mu s)^q} = \int_0^\infty e^{-\mu s v}\phi(v)\,dv$ and $\frac{-1}{\mu}\frac{d}{ds}[e^{-(\mu s)^q}] = \int_0^\infty ve^{-\mu s v}\phi(v)\,dv$, and take the probability density function $\psi(v) = \frac{1}{q}\phi(v^{-\frac{1}{q}})v^{-1-\frac{1}{q}}$, we get

$$\begin{aligned}
J_1 &= \int_0^\infty q(\mu s)^{q-1}e^{-(\mu s)^q}T(s^q)[x_0 + g(x)]\,ds \\
&= \int_0^\infty \int_0^\infty ve^{-\mu s v}\phi(v)\,dv\ T(s^q)[x_0 + g(x)]\,ds \\
&= \int_0^\infty e^{-\mu s}\int_0^\infty \phi(v)T(\frac{s^q}{v^q})\,dv\ [x_0 + g(x)]\,ds \\
&= \int_0^\infty e^{-\mu s}\int_0^\infty \psi(v)T(s^q v)\,dv\ [x_0 + g(x)]\,ds \\
&= \int_0^\infty e^{-\mu s}T_q(s)[x_0 + g(x)]\,ds,
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= \int_0^\infty e^{-\mu^q s}T(s)L\{f(t,x(t)) + Bu(t)\}\,ds \\
&= \int_0^\infty \int_0^\infty qs^{q-1}e^{-(\mu s)^q}T(s^q)e^{-\mu r}L\{f(t,x(t)) + Bu(t)\}\,drds \\
&= q\int_0^\infty \int_0^\infty \int_0^\infty s^{q-1}e^{-\mu(sv+r)}\phi(v)T(s^q)[f(r,x(r)) + Bu(r)]\,dvdsdr \\
&= q\int_0^\infty \int_r^\infty e^{-\mu s}(s-r)^{q-1}[\int_0^\infty \frac{1}{v^q}\phi(v)T((\frac{s-r}{v})^q)dv][f(r,x(r)) + Bu(r)]\,dsdr \\
&= \int_0^\infty e^{-\mu s}\int_0^s (s-r)^{q-1}[q\int_0^\infty v\psi(v)T(s-r)^q v)dv][f(r,x(r)) + Bu(r)]\,drds
\end{aligned}$$

Now, applying the inverse Laplace transform we get the desired result. Therefore, we can give the following definition

Definition 4. The mild solution of the control system (1) is given by

$$\begin{aligned}
x(t) &= \Lambda_k(x)(t) + T_q(t)[x_0 + g(x)] \\
&\quad + \int_0^t (t-s)^{q-1}S_q(t-s)[Bu(s) + f(s,x(s))]\,ds, \qquad (5)
\end{aligned}$$

where $T_q(t) = \int_0^\infty \phi(\theta)T(t^q\theta)\,d\theta$, $S_q(t) = q\int_0^\infty \theta\phi(\theta)T(t^q\theta)\,d\theta$, and ϕ is a probability density function which satisfies $\phi \geq 0$, $\theta \in (0,\infty)$, and $\int_0^\infty \phi(\theta)\,d\theta = 1$.

Lemma 2. *[26]* The operators $T_q(t)$ and $S_q(t)$ are linear and bounded on X, that is $\|T_q(t)\| \leq M\|x\|$ and$\|S_q(t)\| \leq \frac{qM}{\Gamma(1+q)}\|x\|$ for all $x \in X$.

Consider the linear fractional control system

$$\begin{cases} D^q x(t) = Ax(t) + Bu(t), \ b-\delta < t \leq b, \\ x(b-\delta) = x_0, \end{cases} \tag{6}$$

where x and u are the state and control processes, respectively and $0 < \delta \leq b$. The mild solution of the linear system (6) is given by

$$x(t) = T_q(t+\delta-b)x_0 + \int_{b-\delta}^{t} (t-s)^{q-1} S_q(t-s)Bu(s)\, ds, \quad b-\delta < t \leq b \tag{7}$$

and its corresponding controllability operator is given by

$$\Pi_{b-\delta}^{b} = \int_{b-\delta}^{b} (b-s)^{q-1} S_q(b-s)BB^* S_q^*(b-s)\, ds,$$

where B^* and S_q^* are the adjoint of B and S_q, respectively.
It is clear that

$$\Pi_{b-\delta}^{b} = \Pi_0^{\delta}.$$

Definition 5. The control system (6) is said to be approximately controllable on $[0,T]$ if, for every $h \in X$, we have $\overline{R(h,T)} = X$, where

$$R(h,T) = \Big\{ x \in X; \ \ x \text{ is a solution of the system (6)} \Big\}$$

In other words, for any given $\epsilon > 0$ the system (6) can be steered from the initial state x_0 to an ϵ-neighborhood of any target state y at terminal time b.

Theorem 1. *[17]* The following statements are equivalent

(i) The control system (6) is approximately controllable on $[b-\delta, b]$.

(ii) For all nonzero $y \in X$, $\left\langle \Pi_{b-\delta}^{b} y, y \right\rangle > 0$.

(iii) For all $y \in X$, $\alpha(\alpha I + \Pi_{b-\delta}^{b})^{-1} y \to 0$ as $\alpha \to 0^+$.

3 Main results

In this section, we impose the needed assumptions and then demonstrate the approximate controllability of the system 1.

(A1) X and U are Banach spaces;

(A2) A is a densely defined closed linear operator on X, generating a compact analytic semigroup $T(t)$, $t \geq 0$;

(A3) B is a bounded linear operator from U to X;

(A4) f is a nonlinear function from $[0,b] \times X$ to X, satisfying

- f is continuous on $[0,b] \times X$;
- f is Lipschitz with respect to x , that is, for all $t \in [0,b]$ and $x, y \in X$,
$$\|f(t,x) - f(t,y)\| \leq M_1(\|x-y\|)$$
for some $M_1 \geq 0$;
- $\|f(t,x(t))\| \leq M_2$ for some $M_2 > 0$;

(A5) g is a continuous function $[0,b]$ to X and there exist $M_3 > 0$ and $M_4 > 0$ such that

(A6) $U_{\text{ad}} = C(0,b;U)$ is the space of all continuous functions from $[0,b]$ to U.

(A7) The kernel $k(s,t)$ is a continuous function, satisfying

$$\lim_{t_2 \to t_1} \int_0^b \|k(t_2,s) - k(t_1,s)\| \, ds = 0.$$

(A8) For all nonzero $y \in X$, $\left\langle \Pi_{b-\delta}^b y, y \right\rangle > 0$.

Lemma 3. *[20]* Under the assumption $(A7)$, Λ_k is a compact operator from $C([0,b],X)$ into $C([0,b],X)$.

Theorem 2. If the assumptions $(A1 - A7)$ hold true, then the fractional control system (1) has a mild solution.

Proof. Define the linear mapping $F : C([0,b],X) \to C([0,b],X)$, by

$$\begin{aligned}(Fx)(t) &= \Lambda_k x(t) + T_q(t)[x_0 + g(x)] \\ &\quad + \int_0^t (t-s)^{q-1} S_q(t-s)[Bv(s) + f(s,x(s))] \, ds, \qquad (8)\end{aligned}$$

$$(9)$$

It is clear that the fixed points of the operator F are mild solutions of the system (1).
Consider the closed convex set

$$B_R = \left\{ x \in C([0,b],X) \ : x(0) = x_0 + g(x) \text{ and } \|\mathrm{x}\|_\infty \leq \mathrm{R} \right\},$$

where R is selected such that

$$\Lambda_k\|R + M[\|x_0\| + M_3] + \frac{M}{\Gamma(1+q)}b^q\{M_2 + \|B\|\|v\|\} < R.$$

Now, for every $x \in B_R$, we have

$$\begin{aligned}
\|(Fx)(t)\| &\leq \|\Lambda_k x(t)\| + \|T_q(t)[x_0 + g(x)]\| \\
&\quad + \| \int_0^t (t-s)^{q-1} S_q(t-s)[Bv(s) + f(s, x(s))]\, ds\|, \qquad (10)\\
&\leq \|\Lambda_k\|R + M\|x_0 + g(x)\| \\
&\quad + q\frac{M}{\Gamma(1+q)} \int_0^t (t-s)^{q-1}[\|f(s, x(s))\| + \|B\|\|v\|]ds \\
&\leq \|\Lambda_k\|R + M[\|x_0\| + M_3] \\
&\quad + \frac{M}{\Gamma(1+q)}b^q\{M_2 + \|B\|\|v\|\}. \qquad (11)
\end{aligned}$$

Therefore, for every $t \in [0, b]$ We get

$$\|(Fx)(t)\| \leq \|\Lambda_k\|R + M[\|x_0\| + M_3] + \frac{M}{\Gamma(1+q)}b^q\{M_2 + \|B\|\|v\|\},$$

which proves that F maps B_R into itself.
Now, we shall prove that F is compact. So, we can write F separately as $F = F_1 + F_2$, where $(F_1x)(t) = \Lambda_k x(t) + T_q(t)[x_0 + g(x)]$ which is clearly compact (see book) and

$$(F_2x)(t) = \int_0^t (t-s)^{q-1} S_q(t-s)[Bv(s) + f(s, x(s))]\, ds.$$

Let $W(t) = \{(F_2x)(t) : x \in B_R\}$. Obviously, $W(0)$ is compact. Hence, fix $0 < t \leq b$. For $0 < \epsilon < t$ and $\tau > 0$, set

$$\begin{aligned}
(F_2^{\tau,\epsilon}x)(t) &= q\int_0^{t-\tau}\int_\epsilon^\infty \theta(t-s)^{q-1}\phi(\theta)T((t-s)^q\theta)[Bu(s) + f(s, x(s))]\, d\theta ds \\
&= qT(\tau^q\epsilon)\int_0^{t-\tau}\int_\epsilon^\infty \theta(t-s)^{q-1}\phi(\theta)[T((t-s)^q\theta) - T(\tau^q\epsilon)]\times \\
&\quad [Bu(s) + f(s, x(s))]\, d\theta ds \\
&\quad := T(\tau^q\epsilon)p(t, \tau) \qquad (12)
\end{aligned}$$

Therefore, since $T(\tau^q\epsilon)$ is compact and $p(t, \tau)$ is bounded on B_R, the set $W_\tau(t) = \{(F_2^{\tau,\epsilon}x)(t) : x \in B_R\}$ is relatively compact in X. Furthermore, for

every $x \in B_R$, we have

$$\begin{aligned}\|(F_2x)(t) - (F_2^{\tau,\epsilon}x)(t)\| &= q\| \int_0^t \int_0^\epsilon \theta(t-s)^{q-1}\phi(\theta)T((t-s)^q\theta)[Bu(s) + f(s,x(s))]\, d\theta\, ds \\ &\quad + \int_{t-\tau}^t \int_\epsilon^\infty \theta(t-s)^{q-1}\phi(\theta)T((t-s)^q\theta)[Bu(s) + f(s,x(s))]\, d\theta ds\| \\ &\leq M\Big[\frac{\tau^q}{\Gamma(1+q)} + b^q \int_0^\epsilon \theta\phi(\theta)\, d\theta\Big] \sup_{x\in B_R, s\in[0,b]} \|Bu(s) + f(s,x(s))\| \\ &\to 0, \text{ as } \tau, \epsilon \to 0. \end{aligned} \tag{13}$$

This proves that for all $t \in [0,b]$, $W(t)$ is relative compact in X.

Next, we prove that $W(t)$ is equicontinuous on $[0,b]$. Let be $x \in B_R$, and $0 \leq t_1 < t_2 \leq b$. Then, we have

$$\begin{aligned}\|(F_2x)(t_2) - (F_2x)(t_1)\| &\leq \| \int_0^{t_1} (t_1-s)^{q-1}[T(t_2-s) - T(t_1-s)][Bu(s) + f(s,x(s))]\, ds\| \\ &\quad + \| \int_0^{t_1} (t_1-s)^{q-1}[(t_2-s)^{q-1} - (t_1-s)^{q-1}][Bu(s) + f(s,x(s))]\, ds\| \\ &\quad + \| \int_{t_1}^{t_2} (t_2-s)^{q-1}T(t_2-s)[Bu(s) + f(s,x(s))]\, ds\| \\ &= J_1 + J_2 + J_3, \end{aligned} \tag{14}$$

where

$$\begin{aligned} J_1 &= \| \int_0^{t_1} (t_1-s)^{q-1}[T(t_2-s) - T(t_1-s)][Bu(s) + f(s,x(s))]\, ds\| \\ J_2 &= \| \int_0^{t_1} (t_1-s)^{q-1}[(t_2-s)^{q-1} - (t_1-s)^{q-1}][Bu(s) + f(s,x(s))]\, ds\| \\ J_3 &= \| \int_{t_1}^{t_2} (t_2-s)^{q-1}T(t_2-s)[Bu(s) + f(s,x(s))]\, ds\|. \end{aligned} \tag{15}$$

By calculating, for $t_1 > 0$ and $\epsilon > 0$ being sufficiently small enough, we get

$$\begin{aligned} J_1 &\leq \| \int_0^{t_1-\epsilon} (t_1-s)^{q-1}[T(t_2-s) - T(t_1-s)][Bu(s) + f(s,x(s))]\, ds\| \\ &\quad + \| \int_{t_1-\epsilon}^{t_1} (t_1-s)^{q-1}[T(t_2-s) - T(t_1-s)][Bu(s) + f(s,x(s))]\, ds\| \\ &\leq \frac{t_1^q - \epsilon^q}{q}[\|B\|\|u\| + M_1] \sup_{s\in[0,t_1-\epsilon]} \|T(t_2-s) - T(t_1-s)\| \\ &\quad + 2\frac{\epsilon^q M}{\Gamma(1+q)}[\|B\|\|u\| + M_1]. \end{aligned} \tag{16}$$

Since $T(t)$ for $t > 0$ is continuous in the uniform operator topology, it can be obtained that J_1 goes to 0 as $t_2 \to t_1$, $\epsilon \to 0$, independently of x. Similarly, for $t_1 > 0$, we have

$$J_2 \leq \frac{M}{\Gamma(1+q)} \|t_2^q - t_1^q - (t_2 - t_1)^q [\|B\| \|u\| + M_1]. \tag{17}$$

Clearly, J_2 tends to 0 as $t_2 \to t_1$. Next,

$$J_3 \leq \frac{M}{\Gamma(1+q)} (t_2 - t_1)^q [\|B\| \|u\| + M_1],$$

which is obviously tends to 0 as $t_2 \to t_1$. Now, for $t_1 = 0$ and $0 < t_2 \leq b$, it is clear that $J_1, J_2, J_3 \to 0$ as $t_2 \to t_1$. Thus, $\|(F_2 x)(t_2) - (F_2 x)(t_1)\| \to 0$ as $t_2 \to t_1$ independently of $x \in B_R$. Hence, by the Ascoli-Arzela theorem, F_2 is compact. Therefore, $F = F_1 + F_2$ is compact. Then, by the Schauder's fixed point theorem, F has a fixed point which is a mild solution of the fractional control system (1). This completes the proof. □

Theorem 3. Under the assumptions (A1-A8), the fractional control system is approximately controllable for $b > 0$.

Proof. Let $\epsilon > 0$ and $0 < \delta < b$. Consider arbitrary $x_0 \in X$ and $h \in X$. Take any $u \in U_{\text{ad}}$ and the corresponding mild solution of the control system (1). For $\lambda > 0$, define a control $u_{\lambda,\delta}$ by

$$u_{\lambda,\delta}(t) = \begin{cases} u(t), \ 0 \leq t \leq b - \delta, \\ B^* T^*(b-t) \big(\lambda I + \Pi_0^\delta\big)^{-1} \big(h - T(\delta) x(b-\delta)\big), b - \delta < t \leq b, \end{cases} \tag{18}$$

It obvious that $u_{\lambda,\delta} \in U_{\text{ad}}$. Now, the mild solution at terminal time b for the nonlinear system (1) can be written as

$$x(b) = \Lambda_k x(b) + T_q(\delta) x(b-\delta) + \int_{b-\delta}^{b} (b-s)^{q-1} S_q(b-s) [B u_{\lambda,\delta}(s) + f(s, x(s))] \, ds.$$

Similarly, for the mild solution of the linear system (6), we have

$$z(T) = T_q(\delta) z(b-\delta) + \int_{b-\delta}^{b} (b-s)^{q-1} S_q(b-s) [B u_{\lambda,\delta}(s) \, ds.$$

Thus,

$$\begin{aligned} \|x(b) - z(b)\| &\leq \int_{b-\delta}^{b} (b-s)^{q-1} \|S_q(b-s)\| f(s, x(s))\| \, ds \\ &\leq \frac{\delta^q M_2 M}{\Gamma(1+q)} \end{aligned} \tag{19}$$

Therefore,

$$\|x(b) - h\| \leq \frac{\delta^q M_2 M}{\Gamma(1+q)} + \|z(b) - h\|.$$

From Theorem 1, for any $0 < \delta < \min\left\{b, \left(\frac{\epsilon\Gamma(1+q)}{M_2 M}\right)^{\frac{1}{\delta}}\right\}$ the condition $\Pi_0^\delta > 0$ gives that

$$\|z(b) - h\| \leq \frac{\epsilon}{2},$$

for sufficiently small $\lambda > 0$. Finally, for arbitrary $\epsilon > 0$, we obtain

$$\|x(b) - h\| \leq \epsilon.$$

which completes the proof. □

4 Conclusion

In this paper, the approximate controllability problem for semilinear integro-differential fractional systems with nonlocal conditions has been investigated. A set of sufficient conditions for approximate controllability of such control systems is established. The results obtained are beneficial for studying physical problems which are characterized by fractional systems. The future work of this paper is to extend its results to fractional control systems with stochastic inputs.

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