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The Fractional Annihilator Technique for Solving Nonhomogeneous Fractional Linear Differential Equations

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Abstract

In this paper, we apply a fractional annihilator operator to finding a general solution for a constant coefficient nonhomogeneous sequential linear conformable fractional differential equation. The results obtained in this study depend upon the general solution of the homogeneous part with the help of the fractional annihilator operator. The practical validity of this method is demonstrated by some useful illustrative examples.

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1 Introduction

Throughout the last three decades, Fractional derivatives have become a hot topic amongst a large number of researchers. This topic has a wide range of applications in applied sciences such as, physics, militaries, and finances; see the monographs [7, 8] and references therein. In 1965, L'Hopital questioned Leibniz about the meaning of D^n if n were not an integer. Since then, the fractional calculus has attracted the attention of many famous mathematicians such as Laplace, Euler, Abel, Fourier, Laurent, Liouville and Riemann. Actually, several definitions of fractional derivatives have been proposed. The

famous and most applicable definitions are given by Caputo and Riemann-Liouville.

(I) Caputo definition:

$${}^CD_q(f)(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-r)^{n-q-1} f^{(n)}(r) dr, \quad q \in (n-1, n].$$

(II) Riemann-Liouville definition:

$${}^{RL}D_q(f)(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-r)^{n-q-1} f(r) dr, \quad q \in (n-1, n].$$

In spite of the substantial benefits and important utilizations of these two definitions in mathematics and applied sciences, they have been missing the essential properties of ordinary derivatives. For instance: The Riemann-Liouville derivative of nonzero constants is not zero, the quotient rule, the product rule, and the chain rule do not hold, etc.

More recently, Khalil et al. [6] proposed a new definition of fractional derivative that is expressed in limit form and generally satisfied most properties of ordinary derivative, also known as the conformable fractional derivative. This definition is much simpler than that of the existing previous ones and remarkably similar to the one in ordinary case. Based on this definition, a new theory of calculus was presented as the conformable fractional calculus. This theory has been vastly developed and adapted by Abdeljawad [1] and Atangana et al. [3]. Furthermore, a number of investigators have discussed the extensions of a general solution of higher order linear ordinary differential equations to a general solution of sequential linear conformable fractional differential equations. Indeed, they employed the same methods of the ordinary case with a bit adaptation to finding the solution of the conformable fractional differential equations. Unal et al. [9] and Al Horani et al. [2] used the variation of parameters method and Unal et al. [10] utilized the operator method for giving a particular solution of nonhomogeneous sequential linear conformable fractional differential equations. Also, these same fractional differential equations have been solved with the help of undetermined coefficient method in [5].

In this paper, we give the general solution of constant coefficients nonhomogeneous sequential linear conformable fractional differential equations by using the fractional annihilator technique and a fractional exponential function of the corresponding homogeneous part.

This paper is organized as follows: Section 2 gives a well-known definition of the conformable fractional derivative and a set of its frequently usable properties, along with the conformable fractional derivatives of some useful elementary functions. In the third section, a definition of fractional annihilator and the annihilators of some special functions are presented. The general solution

of sequential linear non-homogeneous conformable fractional differential equations is obtained using the fractional annihilator operator in the forth section. In section five, typical examples are given to illustrate the efficiency of the fractional annihilator technique. Our conclusion is drawn in the last section.

2 Preliminaries

In this section, the basic definition of conformable fractional derivative and its essentially usable properties are gathered from conformable fractional calculus. For more details, we refer to [3, 6]. It is worth to be mentioned that all constants in the whole paper are considered to be real and $0 < q \leq 1$.

Definition 1. The conformable fractional derivative of order q of a real function f defined on the interval $[0, \infty)$ is given by

$$D_q(f)(x) = \lim_{\delta \rightarrow 0} \frac{f(x + \delta x^{1-q}) - f(x)}{\delta}, \quad \forall x > 0, \quad q \in (0, 1].$$

If the limit on right hand side exists and is finite, we say that f is q -differentiable at $x > 0$. If f is q -differentiable in some $(0, b)$, $b > 0$, and $\lim_{x \rightarrow 0+} D_q(f)(x)$ exists, then the q -derivative of f at point 0 can be defined as

$$D_q(f)(0) = \lim_{x \rightarrow 0+} D_q(f)(x)$$

Note that if f is differentiable in the ordinary sense, then

$$D_q(f)(x) = x^{1-q} f'(x)$$

Theorem 1. [6] (**basic properties of D_q**) If g and h are two q -differentiable functions at any point $x > 0$, then:

- (i) For all constants p , $D_q(x^p) = px^{p-q}$
- (ii) For all constants c , $D_q(c) = 0$.
- (iii) For all constants a and b , $D_q(ag + bh) = aD_q(g) + bD_q(h)$.
- (iv) $D_q(gh) = D_q(g)h + hD_q(g)$.
- (V) $D_q\left(\frac{g}{h}\right) = \frac{D_q(g)h - hD_q(g)}{h^2}$.

Theorem 2. [6] (**conformable fractional derivative of some elementary functions**) For all constants κ :

- (i) $D_q(\sin \kappa \frac{x^q}{q}) = \kappa \cos \kappa \frac{x^q}{q}$.
- (ii) $D_q(\cos \kappa \frac{x^q}{q}) = -\kappa \sin \kappa \frac{x^q}{q}$.
- (iii) $D_q(e^{\kappa \frac{x^q}{q}}) = \kappa e^{\kappa \frac{x^q}{q}}$.

3 The Fractional Annihilator Operator

In this section, we present the definition of fractional annihilator and the annihilators of some useful functions.

Definition 2. The fractional annihilator of a suitable function f is any fractional differential operator $A(D_q)$ of smallest order which satisfies $A(D_q)f = 0$.

Theorem 3. $(D_q - c)^{n+1}$ annihilates the following functions: $e^{\frac{c}{q}x^q}$, $x^q e^{\frac{c}{q}x^q}$, $x^{2q} e^{\frac{c}{q}x^q}$, $\dots$, $x^{nq} e^{\frac{c}{q}x^q}$, and also annihilates

$$(c_0 + c_1 x^q + c_2 x^{2q} + \dots + c_n x^{nq}) e^{\frac{c}{q}x^q},$$

where $c, c_0, c_1, \dots, c_n$ are arbitrary constants and n is a nonnegative integer.

Proof. This follows directly from Definition 2 and Theorem 1. $\square$

Note that in case that $c = 0$, D_q^{n+1} annihilates the polynomial $c_0 + c_1 x^q + c_2 x^{2q} + \dots + c_n x^{nq}$.

Theorem 4. Let a and b be arbitrary constants. Then ${}^2D_q - 2aD_q + a^2 + b^2$ annihilates the following functions: $e^{\frac{a}{q}x^q} \cos \frac{b}{q}x^q$ and $e^{\frac{a}{q}x^q} \sin \frac{b}{q}x^q$ also annihilates

$$e^{\frac{a}{q}x^q} (a_0 \cos \frac{b}{q}x^q + b_0 \sin \frac{b}{q}x^q), \quad a_0, b_0 \in \mathbb{R}.$$

In particular, ${}^2D_q + b^2$ annihilates $\cos \frac{b}{q}x^q$.

Proof. This follows directly from Definition 2 and Theorem 2. $\square$

Theorem 5. Let a and b be arbitrary constants. Then $({}^2D_q - 2aD_q + a^2 + b^2)^{n+1}$ annihilates the following functions: $e^{\frac{a}{q}x^q} \cos \frac{b}{q}x^q$, $x^q e^{\frac{a}{q}x^q} \cos \frac{b}{q}x^q$, $x^{2q} e^{\frac{a}{q}x^q} \cos \frac{b}{q}x^q, \dots$, $x^{nq} e^{\frac{a}{q}x^q} \cos \frac{b}{q}x^q$ and hence, it annihilates

$$(a_0 + a_1 x^q + \dots + a_n x^{nq}) e^{\frac{a}{q}x^q} \cos \frac{b}{q}x^q + (b_0 + b_1 x^q + \dots + b_n x^{nq}) e^{\frac{a}{q}x^q} \sin \frac{b}{q}x^q,$$

for any constant $a_0, a_1, \dots, a_n, b_0, b_1, \dots, b_n$. and a nonnegative integer n .

Proof. This follows directly from Definition 2 and Theorem 2. $\square$

4 Nonhomogeneous Conformable Fractional Differential Equations

In this section, we state the basic general form of the target fractional differential equations and find the general solution by using the fractional annihilator operator that is defined in previous section.

Consider the sequential linear nonhomogeneous conformable fractional differential equation

$${}^nD_q y + c_1 {}^{n-1}D_q y + \cdots + c_{n-2} {}^2D_q y + c_{n-1} D_q y + c_n y = G(x) \quad (1)$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants, n is a nonnegative integer, ${}^nD_q y = \overbrace{D_q D_q \cdots D_q}^{n \text{ times}} y$, and $G(x)$ is restricted to be one of the following forms:

(I)

$$G(x) = \begin{cases} Ae^{\frac{a}{q}x^q}, \\ Ax^{aq}, \\ A \cos \frac{a}{q}x^q + B \sin \frac{b}{q}x^q. \end{cases} \quad \forall a, b, A, B \in \mathbb{R}$$

(II) Sums and products of functions given in (I).

In operator form, rewrite Eq (1) as

$$P(D_q)y = G(x), \quad (2)$$

where

$$P(D_q) = {}^nD_q + c_1 {}^{n-1}D_q + \cdots + c_{n-2} {}^2D_q + c_{n-1} D_q + c_n, \quad (3)$$

which is called a polynomial fractional differential operator.

Denote by

$$P(r) = r^n + c_1 r^{n-1} + \cdots + c_{n-2} r^2 + c_{n-1} r + c_n, \quad (4)$$

the auxiliary polynomial associated with the polynomial fractional differential operator (3). The polynomial equation $P(r) = 0$ is called the auxiliary (characteristic) equation.

In the homogeneous case where $G(x) = 0$, Emrah nal et al. [9] obtained the general solution as shown in the following theorem.

Theorem 6. Let $\{r_i; i = 1 \cdots m_1\}$ be m_1 distinct real roots with multiplicity $\{\omega_i; i = 1 \cdots m_1\}$ and $\{\alpha_j + i\beta_j, \alpha_j - i\beta_j; j = 1 \cdots m_2\}$ be $2m_2$ distinct complex roots with multiplicity $\{\sigma_j; j = 1 \cdots m_2\}$ of the characteristic equation so that

$\sum_{k=1}^{m_1} \omega_k + 2 \sum_{k=1}^{m_2} \sigma_k = n$. Then, the general solution to the corresponding homogeneous part of Eq (1) is the linear combination of the following sets:

$$\left\{ \left(\frac{x^q}{q} \right)^l e^{\frac{r_m}{q} x^q}; m = 1 \cdots m_1, \quad l = 0 \cdots \omega_m - 1 \right\}$$

and

$$\left\{ \left(\frac{x^q}{q} \right)^l e^{\frac{\alpha_m}{q} x^q} \cos \left(\frac{\beta_m}{q} x^q \right), \left(\frac{x^q}{q} \right)^l e^{\frac{\alpha_m}{q} x^q} \sin \left(\frac{\beta_m}{q} x^q \right); m = 1 \cdots m_2, \quad l = 0 \cdots \sigma_m - 1 \right\}$$

Proof. This theorem is a restatement of [9, Theorem 3.1] and [4, Theorem 4.5]. $\square$

In this work, we focus on the general solution to the nonhomogeneous case. To solve the equation (1) by using annihilator method, we seek a fractional differential annihilator of $G(x)$, $A(D_q)$, so that $A(D_q)(G) = 0$. Now, we have

$$P(D_q)y = G(x), \quad (5)$$

$$A(D_q)G = 0. \quad (6)$$

By operating on (5) with $A(D_q)$, and using (6), we obtain a new sequential linear homogeneous conformable fractional differential equation

$$A(D_q)P(D_q)y = 0 \quad (7)$$

which, consequently, can be solved using a fractional exponential function, see [9, Theorem 3.1]. Clearly, any solution to Eq (5) is also a solution to Eq (7). Therefore, the general solution to Eq (7) must contain the general solution of the corresponding homogeneous part of Eq (5), since $P(D_q)$ is a part of the overall fractional differential operator in Eq (7). As a consequence, in selecting the trial solution for Eq (5), we must select merely that part of the general solution to Eq (7) that does not involve the general solution of the corresponding homogeneous part of Eq (5). To clarify these ideas, we give several examples in next section.

5 Examples

In this section, we give three typical examples which cover all the types of $G(x)$.

Example 1. [10] Use the annihilator technique to solve

$${}^2D_q y + 4D_q y + 3y = 2x^{2q} + x^q - 3. \quad (8)$$

Solution The general solution y_h of the corresponding homogeneous part of the given equation is

$$y_h = c_1 e^{-\frac{1}{q}x^q} + c_2 e^{-\frac{3}{q}x^q}. \quad (9)$$

Since ${}^3D_q(2x^{2q} + x^q - 3) = 0$, it follows that $A(D_q) = {}^3D_q$. Now, operating on Eq (8) with $A(D_q)$ yields

$${}^5D_q y + 4{}^4D_q y + 3{}^3D_q y = 0. \quad (10)$$

So, the characteristic equation is $r^5 + 4r^4 + 3r^3 = 0$ which has the three roots $r = -1$, $r = -3$, and $r = 0$ with multiplicity 3. Then, by Theorem 6 the general solution of the Eq (10) is

$$\begin{aligned} y(x) &= c_1 e^{-\frac{1}{q}x^q} + c_2 e^{-\frac{3}{q}x^q} + c_3 + c_4 \frac{x^q}{q} + c_5 \frac{x^{2q}}{q^2} \\ &= y_h + c_3 + c_4 \frac{x^q}{q} + c_5 \frac{x^{2q}}{q^2} \\ &= y_h + y_p \end{aligned} \quad (11)$$

Hence, an appropriate trial solution for the given Fractional differential equation (8) is $y_p(x) = c_3 + c_4 \frac{x^q}{q} + c_5 \frac{x^{2q}}{q^2}$, where the constants c_3 , c_4 , and c_5 can be calculated by substituting in the equation (8). After substituting y_p into (8), we obtain

$$2c_5 + 4c_4 + \frac{8c_5}{q}x^q + 3c_3 + \frac{3c_4}{q}x^q + \frac{3c_5}{q^2}x^{2q} = 2x^{2q} + x^q - 3.$$

By equating the coefficients of x^{2q} , x^q , and the constant term, we get

$$\begin{aligned} 2c_5 + 4c_4 + 3c_3 &= -3 \\ \frac{8c_5}{q} + \frac{3c_4}{q} &= 1 \\ \frac{3c_5}{q^2} &= 2. \end{aligned}$$

Solving this system for c_3 , c_4 , and c_5 , one can easily find that $c_3 = \frac{-27-12q+52q^2}{27}$, $c_4 = \frac{3-16q}{9}q$, and $c_5 = \frac{2}{3}q^2$. Therefore the particular solution to (10) is

$$y_p(x) = \frac{-27-12q+52q^2}{27} + \frac{3-16q}{9}x^q + \frac{2}{3}x^{2q},$$

which coincide with the particular solution given in [9, Example 4.1] by using the method of variation of parameters, and also coincide with the particular solution given in [10, Example 3.2] by using the operator method.

Example 2. Use the annihilator technique to solve

$${}^2D_{\frac{1}{2}}y - 3y = \cos 2x^{\frac{1}{2}}. \quad (12)$$

Solution The general solution y_h of the corresponding homogeneous part of the given equation is

$$y_h = c_1 e^{-2\sqrt{3}x^{\frac{1}{2}}} + c_2 e^{2\sqrt{3}x^{\frac{1}{2}}}. \quad (13)$$

Since $({}^2D_{\frac{1}{2}} + 1)\cos 2x^{\frac{1}{2}} = 0$, it follows that $A(D_{\frac{1}{2}}) = {}^2D_{\frac{1}{2}} + 1$. Now, operating at Eq (12) with $A(D_{\frac{1}{2}})$ yields

$$({}^2D_{\frac{1}{2}} + 1)({}^2D_{\frac{1}{2}} - 3)y = 0. \quad (14)$$

So, the characteristic equation is $(r^2 + 1)(r^2 - 3) = 0$ which has four distinct roots $-\sqrt{3}$, $\sqrt{3}$, $r = -i$, and $r = i$. Then, by Theorem 6 the general solution of the Eq (14) is

$$\begin{aligned} y(x) &= c_1 e^{-2\sqrt{3}x^{\frac{1}{2}}} + c_2 e^{2\sqrt{3}x^{\frac{1}{2}}} + c_3 \cos 2x^{\frac{1}{2}} + c_4 \sin 2x^{\frac{1}{2}} \\ &= y_h + c_3 \cos 2x^{\frac{1}{2}} + c_4 \sin 2x^{\frac{1}{2}} \\ &= y_h + y_p. \end{aligned} \quad (15)$$

Hence, an appropriate trial solution for the given fractional differential equation (12) is $y_p(x) = c_3 \cos 2x^{\frac{1}{2}} + c_4 \sin 2x^{\frac{1}{2}}$, where the constants c_3 , and c_4 can be calculated by substituting into equation (12). After substituting y_p into (12), we obtain

$$-4c_3 \cos 2x^{\frac{1}{2}} + -4c_4 \sin 2x^{\frac{1}{2}} = \cos 2x^{\frac{1}{2}}.$$

By equating coefficients of $\cos 2x^{\frac{1}{2}}$, and $\sin 2x^{\frac{1}{2}}$, we get $c_3 = -\frac{1}{4}$ and $c_4 = 0$. Consequently, the particular solution to (12) is $y_p(x) = -\frac{1}{4} \cos 2x^{\frac{1}{2}}$.

Example 3. Use the annihilator technique to solve

$$({}^2D_{\frac{1}{3}} - 3)^2 y = 2e^{9x^{\frac{1}{3}}}. \quad (16)$$

Solution The general solution y_h of the corresponding homogeneous part of the given equation is

$$y_h = c_1 e^{9x^{\frac{1}{3}}} + c_2 3x^{\frac{1}{3}} e^{9x^{\frac{1}{3}}}. \quad (17)$$

Since $(D_{\frac{1}{3}} - 3)^2 2e^{9x^{\frac{1}{3}}} = 0$, it follows that $A(D_{\frac{1}{3}}) = (D_{\frac{1}{3}} - 3)^2$. Now, operating at Eq (16) with $A(D_{\frac{1}{3}})$ yields

$$(D_{\frac{1}{3}} - 3)^4 y = 0. \quad (18)$$

So, the characteristic equation is $(r - 3)^4 = 0$ which has only one root $r = 3$ with multiplicity 4. Then, by Theorem 6 the general solution of the Eq (18) is

$$\begin{aligned} y(x) &= c_1 e^{9x^{\frac{1}{3}}} + c_2 3x^{\frac{1}{3}} e^{9x^{\frac{1}{3}}} + c_3 9x^{\frac{2}{3}} e^{9x^{\frac{1}{3}}} + c_4 27x e^{9x^{\frac{1}{3}}} \\ &= y_h + c_3 9x^{\frac{2}{3}} e^{9x^{\frac{1}{3}}} + c_4 27x e^{9x^{\frac{1}{3}}} \\ &= y_h + y_p. \end{aligned} \quad (19)$$

Hence, an appropriate trial solution for the given fractional differential equation (16) is $y_p(x) = c_3 9x^{\frac{2}{3}} e^{9x^{\frac{1}{3}}} + c_4 27x e^{9x^{\frac{1}{3}}}$, where the constants c_3 , and c_4 can be calculated by substituting into equation (16). After substituting y_p into (16), we obtain

$$\begin{aligned} &2c_3 e^{9x^{\frac{1}{3}}} + 36c_3 x^{\frac{1}{3}} e^{9x^{\frac{1}{3}}} + 27(3)c_3 x^{\frac{2}{3}} e^{9x^{\frac{1}{3}}} + 18c_4 x^{\frac{1}{3}} e^{9x^{\frac{1}{3}}} + 27(6)c_4 x^{\frac{2}{3}} e^{9x^{\frac{1}{3}}} \\ &+ 27(9)c_4 x e^{9x^{\frac{1}{3}}} - 36c_3 x^{\frac{1}{3}} e^{9x^{\frac{1}{3}}} - 27(6)c_3 x^{\frac{2}{3}} e^{9x^{\frac{1}{3}}} - 27(6)c_4 x^{\frac{2}{3}} e^{9x^{\frac{1}{3}}} - 27(18)c_4 x e^{9x^{\frac{1}{3}}} \\ &+ 27(3)c_3 x^{\frac{1}{3}} e^{9x^{\frac{1}{3}}} + 27(9)c_4 c_3 x e^{9x^{\frac{1}{3}}} = 2e^{9x^{\frac{1}{3}}}. \end{aligned} \quad (20)$$

By equating coefficients of $e^{9x^{\frac{1}{3}}}$, $x^{\frac{1}{3}} e^{9x^{\frac{1}{3}}}$, $x^{\frac{2}{3}} e^{9x^{\frac{1}{3}}}$, and $x e^{9x^{\frac{1}{3}}}$, we get $c_3 = 1$ and $c_4 = 0$. Consequently, the particular solution to (16) is $y_p(x) = 9x^{\frac{2}{3}} e^{9x^{\frac{1}{3}}}$.

6 Conclusion

In this paper we studied the general solution of the sequential linear nonhomogeneous conformable fractional differential equations. It is known that this solution is split into two essential parts which are the general solution of its corresponding homogeneous part and a particular solution of itself. Based on the annihilator technique, we transferred the nonhomogeneous equation into homogeneous equation and from the general solution of homogeneous equation we extracted a particular solution to the nonhomogeneous one. This transformation is applicable only if the function $G(x)$ in equation (1) has an annihilator. The reliable impact of this technique was illustrated by some typical examples. It is observed that the results obtainable with conformable fractional differential equations are quite similar to that which obtainable with ordinary differential equations.

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